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Half-Range Completeness for the Fokker–Planck Equation

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Half-range completeness theorems are proved for eigenfunctions associated to the one-dimensional Fokker–Planck equation in a semi-infinite medium. Existence and uniqueness results for perfectly absorbing, partially absorbing, and purely specularly reflecting boundary conditions are deduced for the stationary and time-dependent problems. Similar results are obtained for a slab geometry.

KEY WORDS: Fokker–Planck equation (stationary and time dependent); boundary conditions (perfectly absorbing, partially absorbing, purely specularly reflecting); half-range completeness; semi-infinite medium; slab geometry.

1. INTRODUCTION

The problem of determining the distribution function for a Brownian particle in the one-particle phase-space when an absorbing boundary is present goes back to Wang and Uhlenbeck,[1] who acknowledged the difficulty of obtaining a rigorous result and proposed a more straightforward approach.

Burschka and Titulaer[2,5] and Harris[4–6] attacked the problem anew, in connection with the kinetic boundary layer solution of the Fokker–Planck equation (FPE) for several geometries and boundary conditions (BC). The practical interest of the problem comes from the theory of reaction rates and diffusion-controlled reactions, where the FPE sometimes provides a more accurate description than the commonly used diffusion

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equation (DE).[7] Indeed, the diffusion description is limited to the coordinate space and is supposed to be valid only outside the boundary layer regions. Moreover, the BC for the DE cannot be strictly correct, as they cannot distinguish between incident and emergent particles. Therefore, a more complete description, which takes into account the full position–velocity dependence is provided by the FPE, for which purely absorbing[2] or mixed (specular plus diffuse reflection)[3] BC have been imposed. But, as already pointed out in Ref. 1, this turns out to be a delicate problem which has until now resisted attempts at rigorous solution. The main reason for this seems to be the lack of a half-range completeness theorem for the FPE. Instead of this, Burschka and Titulaer[2,3] use numerical methods based on the conjecture that such a theorem is valid, while Harris[4–6] adopts a more ad hoc procedure based on the ansatz that the solution has an explicit half-range character and even a given form, obviously satisfying the BC.

The aim of this paper is to provide the missing half-range completeness theorem for the FPE and to prove existence and uniqueness for various BC.

Theorems of this type have been known for some years for the neutron transport equation[8–10] and for BGK model,[11,12] where special explicit techniques were available. A unified abstract approach to a general "forward–backward" evolution equation was given by Beals[13] but was applied incorrectly to the half-range completeness problem[14] corresponding to perfectly absorbing BC. The authors are indebted to the referee who prevented them from repeating that error here. We apply the method of Ref. 13 with more care and extend it to other BC, to obtain the physically reasonable and mathematically correct results for the FPE. The case of a semi-infinite medium is considered in detail, while the slab geometry is discussed in the last section.

Besides providing a test for the approximations used in previous approaches, the half-range completeness theorem would permit, in principle, the determination of the exact boundary layer solution for the problem at hand and, conseqently, funish a deeper understanding of the diffusion approximation for the FPE.

2. STATEMENT OF THE PROBLEM

The steady one-dimensional Brownian motion of a classical particle of mass m in an isotropic fluid, in thermal equilibrium at the temperature T, can be described by the stationary FPE

$$v \frac{\partial}{\partial x} \psi(x,v) = \left(\gamma \frac{kT}{m} \frac{\partial^2}{\partial v^2} + \gamma \frac{\partial}{\partial v} v\right)\psi(x,v) + \mathscr{S}(x,v) \tag{1}$$

where $\mathscr{S}(x, v)$ is the source term. The range of the velocity v is $\mathbb{R}$, while the position x will be restricted to the right half-axis $x > 0$. For simplicity, and without loss of generality, we shall take the friction coefficient $\gamma = 1$ and $kT/m = 1$. At the wall $x = 0$ we impose one of a family of boundary conditions

$$\psi(0, v) = \alpha\psi(0, -v), \qquad v > 0 \tag{2}$$

where α is a parameter, $0 \leqslant \alpha \leqslant 1$. The extreme case $\alpha = 0$ is the perfectly absorbing boundary,(2) while $\alpha = 1$ describes pure specular reflection.(3) These conditions must be supplemented by a condition far from the boundary $x = 0$. For a source which decays as $x \to \infty$, one expects a solution which is close to the spatially homogeneous solution, i.e.,

$$\lim_{x \to \infty} \psi(x, v) = ce^{-v^2/2} \tag{3}$$

where c is a constant which should be determined from the source term and the BC at $x = 0$.

We shall give this problem a more precise formulation (by specifying appropriate function spaces to which the source term and the solution should belong) and show that it has a unique solution when $0 \leqslant \alpha < 1$. When $\alpha = 1$ the problem has a solution if and only if the source term satisfies a (single) linear constraint, in which case the solution is unique up to the addition of an equilibrium solution $de^{-v^2/2}$, d constant.

After the transformation $\psi = fe^{-v^2/2}$, Eq. (1) reads

$$T\frac{\partial f}{\partial x} + Af = s \tag{4}$$

where $(Tf)(x, v) = vf(x, v)$, $Af(x, v) = -\partial^2 f(x, v)/\partial v^2 + v\,\partial f(x, v)/\partial v$, and $s(x, v) = \mathscr{S}(x, v)e^{v^2/2}$. The boundary conditions (2), (3) take the form

$$f(0, v) = \alpha f(0, -v), \qquad v > 0 \tag{5}$$

and

$$\lim_{x \to \infty} f(x, v) = c \tag{6}$$

It is convenient to replace f by $f - c$; moreover we may easily consider inhomogeneous BC. Then Eq. (4) is unchanged, while (5), (6) become

$$f(0, v) - \alpha f(0, -v) = (\alpha - 1)c + g(v), \qquad v > 0 \tag{7}$$

and

$$\lim_{x \to \infty} f(x, v) = 0 \tag{8}$$

Again, c is a constant to be determined, and g is a given function.

For a precise formulation we introduce the Hilbert spaces of real-valued functions on the line $\mathbb{R}$:

$$\mathscr{H} = L^2(\mathbb{R}; d\sigma), \qquad d\sigma(v) = \frac{1}{\sqrt{2\pi}} e^{-v^2/2}\, dv \tag{9}$$

$$\mathscr{H}_T = L^2(\mathbb{R}; |v|\, d\sigma) \tag{10}$$

with their corresponding inner products and norms

$$(u, w) = \int_{-\infty}^{\infty} u(v) w(v)\, d\sigma(v), \qquad |u| = (u, u)^{1/2} \tag{11}$$

$$(u, w)_T = \int_{-\infty}^{\infty} |v| u(v) w(v)\, d\sigma(v), \qquad |u|_T = (u, u)_T^{1/2} \tag{12}$$

We consider also spaces of functions of $x \geqslant 0$ with values in $\mathscr{H}_T$, particularly

$$L^2(\mathbb{R}_+; \mathscr{H}_T), \qquad L^1(\mathbb{R}_+; \mathscr{H}_T), \qquad C_0(\mathbb{R}_+; \mathscr{H}_T) \tag{13}$$

The first of these three spaces can be identified with the function space

$$L^2(\mathbb{R}_+ \times \mathbb{R}, |v|\, dx\, d\sigma(v)) \tag{14}$$

and the second with the space of functions f with

$$\int_0^{\infty} |f(x, \cdot)|_T\, dx = \int_0^{\infty} \left[\int_{-\infty}^{\infty} f(x, v)^2 |v|\, d\sigma(v) \right]^{1/2} dx < \infty \tag{15}$$

The third space in (13) is the space of functions continuous for $0 \leqslant x < \infty$ with values in $\mathscr{H}_T$, and vanishing at ∞:

$$\lim_{y \to x} \int_{-\infty}^{\infty} [f(y, v) - f(x, v)]^2 |v|\, d\sigma(v) = 0 \tag{16}$$

$$\lim_{x \to \infty} \int_{-\infty}^{\infty} f(x, v)^2 |v|\, d\sigma(v) = 0 \tag{17}$$

For a function $f \in C_0(\mathbb{R}_+; \mathscr{H}_T)$ the boundary condition (7) has a sense and (17) may be taken as a precise form of the condition (8). We consider Eq. (4) in the sense of distributions.

The following result is proved in Section 6 below, after preliminary results in Sections 3–5.

Theorem 1. Suppose g is in $\mathscr{H}_T$ and suppose $v^{-1} s(x, v) = s_1(x, v)$ is such that

$$s_1 \in L^1(\mathbb{R}_+; \mathscr{H}_T) \cap L^2(\mathbb{R}_+; \mathscr{H}_T), \qquad x s_1 \in L^1(\mathbb{R}_+; \mathscr{H}_T)$$

When $0 \leqslant \alpha <$ the problem (4), (7) has a unique solution f which belongs to $C_0(\mathbb{R}_+;\mathscr{H}_T)$ and such that $\partial f/\partial v$ belongs to $L^2(\mathbb{R}_+;\mathscr{H})$. When $\alpha = 1$ the solution exists if and only if s satisfies a certain linear constraint, and again it is unique.

[Note that when $\alpha = 1$, in the formulation (5), (6) a solution is only unique up to addition of a constant.]

The function spaces here are not chosen arbitrarily; they occur naturally in the variational approach, as in Ref. 13.

3. A FORMAL SOLUTION; THE HALF-RANGE COMPLETENESS QUESTION

In connection with the operator occurring in Eq. (4) it is natural to consider the "eigenvalue problem"

$$Au_n(v) = \lambda_n v u_n(v) = \lambda_n T u_n(v) \tag{18}$$

in which we note the interplay between λ_n and the multiplication operator T.

In his development of the full-range theory for the FPE, Pagani[15] calculates explicitly the eigenvalues and eigenfunctions. The eigenvalues are $\lambda_0 = 0$ and $\lambda_{\pm n} = \pm\sqrt{n}$, $n = 1, 2, 3, \ldots$, while the eigenfunctions are $u_0 = C_0$ and

$$u_{\pm n}(v) = C_n e^{\pm v\sqrt{n}} H_n\left(\frac{v}{\sqrt{2}} - \sqrt{2n}\right), \qquad n = 1, 2, \ldots \tag{19}$$

where the H_n are the Hermite polynomials, and the constants C_n will be chosen later. As we shall see, these functions *together with* v are complete in the space $\mathscr{H}_T$ of (10).

Corresponding to a source term s we consider $s_1 = v^{-1}s$ and assume s_1 belongs to $L^1(\mathbb{R}_+;\mathscr{H}_T)$. Then s_1 has an expansion

$$s_1(x, v) = \beta_0(x) + \beta_1(x)v + \sum b_n(x)u_n(v) \tag{20}$$

We look for a solution of (4), (7), (17) which belongs to $C_0(\mathbb{R}_+;\mathscr{H}_T)$. It should have an expansion

$$f(x, v) = \alpha_0(x) + \alpha_1(x)v + \sum a_n(x)u_n(x) \tag{21}$$

Now in view of (18) and the fact that $A1 = 0$ and $Av = v$, we see that (4), (20), (21) yield the system of equations

$$\begin{gathered} \alpha_1'(x) = \beta_1(x), \qquad \alpha_0'(x) + \alpha_1(x) = \beta_0(x) \\ a_n'(x) + \lambda_n a_n(x) = b_n(x) \end{gathered} \tag{22}$$

Taking into account the boundary condition (17) we see that α_0, α_1, and a_n

for $n < 0$ (i.e., $\lambda_n < 0$) are uniquely determined by (22):

$$\alpha_1(x) = -\int_x^\infty \beta_1(y)\,dy \tag{23}$$

$$\alpha_0(x) = \int_x^\infty \left[\alpha_1(y) - \beta_0(y)\right] dy \tag{24}$$

$$a_n(x) = -\int_x^\infty e^{\lambda_n(y-x)} b_n(y)\,dy, \qquad \lambda_n < 0 \tag{25}$$

For $n > 0$, on the other hand, we obtain

$$a_n(x) = a_n e^{-\lambda_n x} + \int_0^x e^{\lambda_n(y-x)} b_n(y)\,dy, \qquad \lambda_n > 0 \tag{26}$$

where the constants a_n, $n > 0$, remain to be determined. The boundary condition (7) requires

$$\sum_{n>0} a_n\left[u_n(v) - \alpha u_n(-v)\right] = c(\alpha - 1) + h(v), \qquad v > 0 \tag{27}$$

where $h(v)$ is the function

$$g(v) - \alpha_0(0)(1-\alpha) - \alpha_1(0)(1+\alpha)v - \sum_{n<0} a_n(0)\left[u_n(v) - \alpha u_n(-v)\right] \tag{28}$$

and c is a constant to be determined. Thus our problem has (formally) been reduced to solving (27) for a known function h and an unknown constant c. The question of the existence and uniqueness for $0 \leqslant \alpha < 1$ is thus the following *half-range completeness question*: are the functions

$$1, \qquad u_n(v) - \alpha u_n(-v), \qquad n > 0 \tag{29}$$

independent and complete among the functions defined on $\{v > 0\}$? A more precise version is: does every function $h \in L^2(\mathbb{R}_+, v\,d\sigma(v))$ have a unique expansion in the functions (29), with this expansion converging in this Hilbert space? We shall show that the answer is yes and that the procedure above can indeed be used to prove Theorem 1.

4. THE EXISTENCE OF EIGENFUNCTIONS; THE OPERATOR S AND THE SPACE $\mathscr{H}_S$

The operator $A = -(\partial/\partial v)^2 + v\,\partial/\partial v$ is formally self-adjoint in the Hilbert space $\mathscr{H}$ introduced above. It has a complete set of eigenfunctions

$$\varphi_n(v) = C_n' H_n(2^{-1/2}v), \qquad n = 0, 1, 2, \ldots \tag{30}$$

Again the H_n are the Hermite polynomials and the corresponding eigenvalues are $\mu_n = n$.[16] We normalize so that

$$(\varphi_n, \varphi_m) = \delta_{nm} \tag{31}$$

In particular

$$\varphi_0(v) = 1, \qquad \varphi_1(v) = v, \qquad \varphi_2(v) = \frac{1}{\sqrt{2}}(v^2 - 1) \tag{32}$$

We would like to consider (18) in the standard eigenvalue form $A^{-1}Tu_n = \lambda_n^{-1}u_n$. Yet the operator A is not invertible; it is convenient then to replace it with the operator

$$A_1 = A + P \tag{33}$$

where P is the orthogonal projection of $\mathscr{H}$ onto the multiples of the function v^2:

$$Pu = (u, v^2)(v^2, v^2)^{-1}v^2 = \tfrac{1}{3}(u, v^2)v^2 \tag{34}$$

Since v^2 is a linear combination of φ_0 and φ_2,

$$A_1\varphi_1 = A\varphi_1 = \varphi_1, \qquad A_1\varphi_n = A\varphi_n = n\varphi_n, \qquad n \geqslant 3 \tag{35}$$

For the time being, we restrict attention to the space $\mathscr{H}_0$ of finite combinations of the φ_n, i.e., the space of polynomial functions of v. This space is invariant for A, A_1, and T. We introduce an inner product

$$(u, w)_A = (A_1u, w) = (Au, w) + (Pu, w) \tag{36}$$

An integration by parts and use of (34) gives

$$(u, w)_A = \left(\frac{du}{dv}, \frac{dw}{dv}\right) + \frac{1}{3}(u, v^2)(w, v^2) \tag{37}$$

In particular, $(u, u)_A = 0$ implies $u = 0$. We define the norm

$$|u|_A = (u, u)_A^{1/2} \tag{38}$$

On the span of $\{\varphi_n;\ n \geqslant 3\}$ we have $|u|_A^2 = (Au, u) \geqslant 3|u|^2$. Since a similar inequality is valid on the span of $\{\varphi_0, \varphi_1, \varphi_2\}$, we obtain an estimate

$$|u| \leqslant C|u|_A \tag{39}$$

Looking at these same two subspaces, we see that A_1 is invertible as an operator in $\mathscr{H}_0$. We let

$$S = A_1^{-1}T \tag{40}$$

Let $\mathscr{H}_A$ denote the completion of $\mathscr{H}_0$ with respect to the norm (38). In view of (39), $\mathscr{H}_A$ may be considered as a subspace of $\mathscr{H}$; in view of (37) it can be shown to consist precisely of those $u \in \mathscr{H}$ such that the (distribution) derivative du/dv is also in $\mathscr{H}$:

$$\mathscr{H}_A = \{u \in \mathscr{H} : du/dv \in \mathscr{H}\} \tag{41}$$

Proposition 1. The operator S extends to a compact self-adjoint operator in $\mathscr{H}_A$.

Proof. S is symmetric on $\mathscr{H}_0$:

$$(Su, w)_A = (Tu, w) = (u, Tw) = (u, Sw)_A \tag{42}$$

An integration by parts gives the identity

$$|Tw|^2 = |w|^2 + 2(Tw, dw/dv) \leqslant |w|^2 + 2|Tw|\,|w|_A \tag{43}$$

From (43), (39), and the inequality $2|st| \leqslant \frac{1}{2}s^2 + 2t^2$ we obtain

$$|Tw| \leqslant C'|w|_A \tag{44}$$

It follows from (42), (44), and (39) that S extends to a bounded self-adjoint operator on $\mathscr{H}_A$. To show that S is compact, it is enough to show that the norm of its restriction to span $\{\varphi_n : n \geqslant m\}$ tends to zero as $m \to \infty$. But for u in this span and $m \geqslant 3$ we obtain from (42) and (44) that

$$|(Su, w)_A| \leqslant |u|\,|Tw| \leqslant C'|u|\,|w|_A \leqslant \frac{1}{\sqrt{m}} C'|u|_A|w|_A \tag{45}$$

since for such u, $|u|_A^2 = (Au, u) \geqslant m|u|^2$.

The identity (42) with $w = Tu$ implies that if $Su = 0$ then $u = 0$. Therefore the compact self-adjoint operator S has a complete set of eigenfunctions in $\mathscr{H}_A$, with nonzero eigenvalues. Consider first the two-dimensional subspace

$$\mathscr{H}_A' = \text{span}\{1, v\} = \text{span}\{\varphi_0, \varphi_1\} \tag{46}$$

Note that $A_1 1 = P1 = v^2/3$ and $A_1 v = Av = v$. Therefore $Sv = A_1^{-1}v^2 = 1$ and $S_1 = A_1^{-1}v = v$, so $\mathscr{H}_A'$ is invariant for S and contains the eigenfunctions

$$\varphi_\pm = C_\pm(\sqrt{3} \pm v) \tag{47}$$

with eigenvalues $\lambda_\pm = \pm\sqrt{3}$; the normalization constant $C_\pm$ will be chosen later. It follows from our computation of $A_1 1$ and $A_1 v$ that the orthogonal complement of $\mathscr{H}_A'$ in $\mathscr{H}_A$ is the space

$$\mathscr{H}_A'' = \{u \in \mathscr{H}_A : (u, v) = (u, v^2) = 0\} \tag{48}$$

Since $\mathscr{H}_A''$ is invariant for the compact self-adjoint operator S, it contains a complete orthogonal set of eigenfunctions $\{u_n\}$. For these eigenfunctions we have

$$Au_n = \lambda_n Tu_n \tag{49}$$

In fact the eigenfunctions clearly satisfy this equation (at least in the sense of distributions) with A replaced by A_1. But since $u \in \mathscr{H}_A''$ implies $(u, v^2) = 0$, we have $A = A_1$ on $\mathscr{H}_A''$. We order the eigenvalues with

$$n\lambda_n > 0, \qquad |\lambda_1| \leqslant |\lambda_2| \leqslant \cdots \tag{50}$$

Let R be the reflection operator:

$$Ru(v) = u(-v), \qquad v \in \mathbb{R} \tag{51}$$

It is easily seen that $AR = RA$, $TR = -RT$, and $u_n = Ru_{-n}$. This argument produces the desired complete set of eigenfunctions and shows that there are infinitely many eigenvalues of each sign; of course these are the eigenfunctions computed by Pagani.[15]

We conclude this section by introducing one more inner product and corresponding norm, and by normalizing the eigenfunctions of S. Corresponding to the self-adjoint operator S in $\mathscr{H}_A$ is the operator

$$|S| = (S^* S)^{1/2} \tag{52}$$

characterized by

$$|S| u_n = |\lambda_n|^{-1} u_n, \qquad |S| \varphi_\pm = |\lambda_\pm|^{-1} \varphi_\pm \tag{53}$$

For $u, v \in \mathscr{H}_A$ we define

$$(u, v)_S = (|S| u, v)_A, \qquad |u|_S = (u, u)_S^{1/2} \tag{54}$$

Finally, we normalize the eigenfunctions by requiring

$$|u_n|_S = 1 = |\varphi_\pm|_S \tag{55}$$

Note that

$$|u|_S^2 = (A_1 |S| u, u) = (A_1 |A_1^{-1} T| u, u) \tag{56}$$

$$|u|_T^2 = (|T| u, u) \tag{57}$$

where $|T| = (T^* T)^{1/2}$ is multiplication by $|v|$. Thus we can hope for a relationship between these norms.

Proposition 2. There is a positive constant m such that for every $u \in \mathscr{H}_0$,

$$m^{-1} |u|_S \leqslant |u|_T \leqslant m |u|_S \tag{58}$$

In particular, (58) means that $\mathscr{H}_T$, which can be taken to be the completion of $\mathscr{H}_0$ with respect to the norm $|\ |_T$, coincides with $\mathscr{H}_S$, the completion of $\mathscr{H}_0$ with respect to $|\ |_S$. Proposition 2 is a necessary but technical point whose proof is indicated rather cryptically in Ref. 13; we give a more detailed proof in the Appendix below.

5. PROJECTIONS; PROOF OF HALF-RANGE COMPLETENESS

To formulate and prove half-range completeness for the eigenfunctions of S, it is convenient to introduce certain orthogonal projections. In $\mathscr{H}_T$ there are natural complementary orthogonal projections Q_+, Q_- defined by

$$Q_+ u(v) = 0, \qquad v < 0, \qquad Q_+ u(v) = u(v), \qquad v > 0 \quad (59)_+$$

$$Q_- u(v) = u(v), \qquad v < 0, \qquad Q_- u(v) = 0, \qquad v > 0 \quad (59)_-$$

Then clearly

$$TQ_+ = Q_+ T, \qquad TQ_- = Q_- T, \qquad |T| = T(Q_+ - Q_-) \tag{60}$$

The analogous projections for the operator S in $\mathscr{H}_S = \mathscr{H}_T$ are the complementary orthogonal projections P_+, P_- defined by

$$P_+ u = (u, \varphi_+)_S \varphi_+ + \sum_{n>0} (u, u_n)_S u_n \quad (61)_+$$

$$P_- u = (u, \varphi_-)_S \varphi_- + \sum_{n<0} (u, u_n)_S u_n \quad (61)_-$$

Then

$$SP_+ = P_+ S, \qquad SP_- = P_- S, \qquad |S| = S(P_+ - P_-) \tag{62}$$

Consider now the half-range completeness question as posed at the end of Section 3, for the case $\alpha = 0$. If we replace the constant function 1 by φ_+, the question is whether Q_+, considered as operating from $P_+(\mathscr{H}_S)$ to $Q_+(\mathscr{H}_T)$, is onto and has a bounded inverse. One may pose the analogous question for Q_- and P_- (indeed it is equivalent, by symmetry) and it is clear that the two questions together are the question whether

$$V = Q_+ P_+ + Q_- P_- \tag{63}$$

is onto and has bounded inverse as operator in $\mathscr{H}_T = \mathscr{H}_S$. Similarly, let R be the reflection operator (50). If again we replace 1 by φ_+, the half-range completeness question for $0 \leqslant \alpha < 1$ can be settled by considering the operator

$$V_\alpha = Q_+(I - \alpha R)P_+ + Q_-(I - \alpha R)P_- \tag{64}$$

Proposition 3. For $0 \leqslant \alpha \leqslant 1$ the operator V_α is one-to-one and onto from $\mathscr{H}_S = \mathscr{H}_T$ to itself, with bounded inverse.

Proof. We use the four identities

$$\begin{aligned} (Q_+ u, P_\pm w)_T &= \pm(Q_+ u, P_\pm w)_S \\ (Q_- u, P_\pm w)_T &= \mp(Q_- u, P_\pm w)_S \end{aligned} \tag{65}$$

For example,

$$\begin{aligned} (Q_+ u, P_- w)_T &= (TQ_+ u, P_- w) = (Q_+ u, TP_- w) = (Q_+ u, A_1 SP_- w) \\ &= -(Q_+ u, A_1 |S| P_- w) = -(Q_+ u, P_- w)_S \end{aligned} \tag{66}$$

Let

$$W = Q_+ P_- + Q_- P_+ \tag{67}$$

Then using (65) we obtain the identity

$$|Vu|_T^2 = |u|_S^2 + |Wu|_T^2 \tag{68}$$

which implies [because of (58)] that

$$|Vu|_T \geqslant \delta |u|_T + |Wu|_T \tag{69}$$

for some $\delta > 0$. It follows that V is one-to-one and has closed range, since convergence of Vw_m implies convergence of w_m. If V is not onto, then there is $w \in \mathscr{H}_T$ which is orthogonal to the range, with $w \neq 0$. But then, using (65), we obtain for all $u \in \mathscr{H}_S$,

$$0 = (Vu, w)_T = (u, V'w)_S, \qquad V' = P_+ Q_+ + P_- Q_- \tag{70}$$

Now V' and $W' = P_+ Q_- + P_- Q_+$ satisfy the identity analogous to (68); thus (70) implies $V'w = 0$ which implies $w = 0$. This proves the assertion for $V = V_0$.

For the remaining cases we note that the reflection operator clearly satisfies

$$RQ_+ = Q_- R, \qquad RQ_- = Q_+ R \tag{71}$$

The discussion in the proof of Proposition 1 implies

$$RP_+ = P_- R, \qquad RP_- = P_+ R$$

Moreover, R is unitary in $\mathscr{H}_S$ and $\mathscr{H}_T$. Then

$$V_\alpha = V - \alpha RW \tag{72}$$

and from (69) we obtain

$$|V_\alpha u|_T \geqslant |Vu|_T - \alpha |Wu|_T \geqslant \delta |u|_T \tag{73}$$

Thus V_α is one-to-one and has closed range. Again, if w is orthogonal to the range of V_α in $\mathscr{H}_T$, then $(V' + \alpha RW')w = 0$ and as before we conclude that $w = 0$ and that V_α is onto with inverse having norm $\leqslant \delta^{-1}$. ■

As noted, we have now solved the half-range completeness problem for $0 \leqslant \alpha \leqslant 1$ with φ_+ in place of 1: in fact since $\{\varphi_+, u_n;\ n > 0\}$ are a complete orthonormal set in $P_+(\mathscr{H}_S)$, we have shown in effect that any $h \in Q_+(\mathscr{H}_T)$ has a unique expansion in the functions $\{Q_+(I - \alpha R)\varphi_+,\ Q_+(I - \alpha R)u_n,\ n > 0\}$ which converges in $\mathscr{H}_T$. Let

$$K_\alpha = \text{closed linear span of } \{Q_+(I - \alpha R)u_n,\ n > 0\} \tag{74}$$

in $Q_+(\mathscr{H}_T)$. Then K_α has one-dimensional complement, and we may replace $Q_+(I - \alpha R)\varphi_+$ in the expansion by some other function $\chi \in Q_+(\mathscr{H}_T)$ if and only if χ is not in K_α. In particular we would like to use $Q_+(1-\alpha R)1=(1-\alpha)Q_+1$. This is not possible when $\alpha=1$. For $0\leqslant\alpha<1$ the question is whether Q_+1 belongs to K_α. Thus the following completes the solution of the half-range completeness question as posed at the end of Section 3.

Proposition 4. Q_+1 is not in K_α, $0\leqslant\alpha\leqslant 1$.

Proof. Suppose $h\in\operatorname{span}\{u_n,\ n>0\}$. Then $P_+h=h$ and so using (65) again we obtain

$$\begin{aligned}(Th,h)&=((Q_+-Q_-)h,P_+h)_T=((Q_++Q_-)h,h)_S\\&=(h,h)_S=|h|_S^2\end{aligned}\tag{75}$$

Similarly $P_-Rh=Rh$, so

$$(TRh,Rh)=-|Rh|_S^2=-|h|_S^2\tag{76}$$

$$(TRh,h)=((Q_++Q_-)Rh,h)_S=(Rh,h)_S=0\tag{77}$$

Moreover, $g=(I-\alpha R)h$ is in $\mathscr{H}_A''$, so

$$(g,v)=(g,A_1 1)=(g,1)_A=0\tag{78}$$

We combine these observations to obtain a positive lower bound on the distance from Q_+g to Q_+1, thereby proving the assertion. In fact

$$\begin{aligned}2|Q_+(g-1)|_T^2&=([Q_++Q_-+Q_+-Q_-]|T|(g-1),g-1)\\&=(|T|(g-1),g-1)+(T(g-1),g-1)\\&=|g-1|_T^2+(Tg,g)-2(g,v)+(v,1)\\&=|g-1|_T^2+(Tg,g)\\&=|g-1|_T^2+(T(h-\alpha Rh),h-\alpha Rh)\\&=|g-1|_T^2+|h|_S^2-\alpha^2|h|_S^2\\&>|g-1|_T^2\geqslant\delta'|g-1|_S^2=\delta'(|g|_S^2+|1|_S^2)\\&\geqslant\delta'|1|_S^2=\delta''>0\end{aligned}\tag{79}$$

6. PROOF OF THEOREM 1

We begin with a resumé of the results of the preceding two sections. The functions $\{\varphi_\pm,u_n\}$ are an orthogonal basis for the space $\mathscr{H}_T=\mathscr{H}_S$ with respect to the inner product (54). Now $\operatorname{span}\{1,v\}=\operatorname{span}\{\varphi_+,\varphi_-\}$. Therefore any element u of $\mathscr{H}_T$ has a unique convergent expansion

$$u=\alpha_0+\alpha_1 v+\sum a_n u_n\tag{80}$$

Moreover

$$|u|_S^2 = |\alpha_0 + \alpha_1 v|_S^2 + \sum a_n^2 \tag{81}$$

and clearly

$$C^{-1}(\alpha_0^2 + \alpha_1^2) \leqslant |\alpha_0 + \alpha_1 v|_S^2 \leqslant C(\alpha_0^2 + \alpha_1^2) \tag{82}$$

Moreover, for $0 \leqslant \alpha < 1$, if h is in $\mathscr{H}_T$, then there is a unique u_α in the closed linear span of $\{1, u_n, n > 0\}$ such that

$$Q_+(u_\alpha - \alpha R u_\alpha) = -Q_+ h \tag{83}$$

The norm of u_α is dominated by that of h.

We can now justify the procedure of Section 3. Suppose first that $s_1 : [0, \infty] \to \mathscr{H}_T$ is continuous, vanishes for large x, and takes values in the subspace spanned by 1, v, and $\{u_n;\ |n| \leqslant N\}$. Then the expansion (20) is finite and the coefficients are continuous and vanish for large x. The functions α_1, α_0, and a_n for $n < 0$ are then determined by (23)–(25). In order to pass to more general functions s_1 by a limiting argument, we need to obtain estimates independent of N and independent of the continuity and vanishing assumptions. Set

$$\begin{aligned} M_x(s_1) &= \int_x^\infty (1+y)|s_1(y,\cdot)|_T\,dy + \left[\int_x^\infty |s_1(y,\cdot)|_T^2\,dy\right]^{1/2} \\ M(s_1) &= M_0(s_1) \end{aligned} \tag{84}$$

In what follows the constant C varies from inequality to inequality but depends only on α. From (23)–(25) and the Cauchy–Schwarz inequality we obtain, using (81), (82), the inequalities

$$|\alpha_1(x)| \leqslant \int_x^\infty |\beta_1(y)|\,dy \leqslant C\int_x^\infty |s_1(y,\cdot)|_T\,dy \tag{85}$$

$$\begin{aligned} |\alpha_0(x)| &\leqslant \int_x^\infty \int_x^\infty |\beta_1(z)|\,dz + \int_x^\infty |\beta_0(y)|\,dy \\ &= \int_x^\infty (z-x)|\beta_1(z)|\,dz + \int_x^\infty |\beta_0(y)|\,dy \\ &\leqslant C\int_x^\infty (1+y)|s_1(y,\cdot)|_T\,dy \end{aligned} \tag{86}$$

$$\begin{aligned} \sum_{n<0} a_n(x)^2 &\leqslant \sum \left(\int_x^\infty e^{2\lambda_n(y-x)}\,dy\right)\int_x^\infty b_n(y)^2\,dy \\ &\leqslant C\int_x^\infty |s_1(y,\cdot)|_T^2\,dy \end{aligned} \tag{87}$$

In particular, the function h defined by (28) satisfies

$$|h|_S \leqslant CM(s_1) \tag{88}$$

The boundary condition (27), which is (83), determines unique constants a_n and c, and we have

$$c^2 + \sum_{n>0} a_n^2 \leqslant C\left[M(s_1)^2 + |g|_T^2\right] \tag{89}$$

The equations (26) then determine functions $a_n(x)$ for $n > 0$. Looking at the two terms separately, splitting the integration into an integral from 0 to $\frac{1}{2}x$ and from $\frac{1}{2}x$ to x, and using the Cauchy–Schwarz inequality, we obtain

$$a_n(x)^2 \leqslant C\left[e^{-2\delta x}a_n^2 + e^{-\delta x}\int_0^{x/2} b_n(y)^2\,dy + \int_{x/2}^{x} b_n(y)^2\,dy\right] \tag{90}$$

where $\delta = \inf|\lambda_n|$. Therefore

$$\sum_{n>0} a_n(x)^2 \leqslant C\left\{e^{-\delta x}\left[M(s_1)^2 + |g|_T^2 + M_x(s_1)^2\right]\right\} \tag{91}$$

Note also that the functions $a_n(x)$ are all continuous. It follows from (85)–(87) and (91) that the series on the right in (21) converges in $\mathscr{H}_T$ for each $x \geqslant 0$ and defines an element f of the space $C_0(\mathbb{R}_+, \mathscr{H}_T)$ of continuous $\mathscr{H}_T$-valued functions vanishing at infinity. Moreover because of the nature of the estimates, we may pass to the limit and obtain f so long as $M(s_1)$ is finite.

Suppose therefore that $M(s_1)$ is finite and that $f \in C_0(\mathbb{R}_+, \mathscr{H}_T)$ has been obtained as above. The individual coefficients satisfy the equations (22) a.e. and are absolutely continuous, so by considering partial sums and passing to the limit we obtain the desired equation

$$v\frac{\partial f}{\partial x} + Af = s \tag{92}$$

in the sense of distributions, where $s = vs_1$. A similar passage to the limit justifies the following calculation:

$$\begin{aligned}
\int_0^\infty \int_{-\infty}^\infty \left[\frac{\partial f}{\partial v}(x,v)\right]^2 d\sigma(v)\,dx &= \int_0^\infty (Af, f)\,dx \\
&= \int_0^\infty (s, f)\,dx - \int_0^\infty \left(v\frac{\partial f}{\partial x}, f\right) dx \\
&= \int_0^\infty (s, f)\,dx - \frac{1}{2}\int_0^\infty \frac{d}{dx}(vf, f)\,dx \\
&= \int_0^\infty (s, f)\,dx + \frac{1}{2}\int_{-\infty}^\infty vf(0,v)^2\,d\sigma(v)
\end{aligned} \tag{93}$$

Taking into account the boundary condition (7),

$$\int_{-\infty}^\infty vf(0,v)^2\,d\sigma(v) = (1-\alpha^2)\int_{v<0} vf(0,v)^2\,d\sigma(v) + \int_{v>0} vg(v)^2\,d\sigma(v) \tag{94}$$

Therefore (93) and our earlier estimates give

$$\int_0^\infty \left|\frac{\partial f}{\partial v}\right|^2 dx \leqslant \int_0^\infty (v s_1, f)\, dx + \int_{v>0} v g(v)^2 \, d\sigma(v)$$
$$\leqslant \int_0^\infty |s_1(x,\cdot)|_T |f(x,\cdot)|_T \, dx + |g|_T^2 \leqslant C\left[M(s_1)^2 + |g|_T^2\right] \quad (95)$$

This completes the proof of Theorem 1, in the cases $0 \leqslant \alpha < 1$.

When $\alpha = 1$ we construct the functions α_1, α_0, a_n, $n < 0$ as before, giving h. Now the necessary and sufficient condition for solvability of the boundary condition (27) is that h lie in $\mathscr{K}_{\alpha=1}$, the closed linear span of $\{Q_+(u_n - Ru_n),\ n > 0\}$. $\mathscr{K}_{\alpha=1}$ has codimension 1 in $Q_+(\mathscr{H}_T)$, so if $\chi \in Q_+(\mathscr{H}_T)$ is chosen orthogonal to $\mathscr{K}_{\alpha=1}$, the condition is

$$\int_{v>0} v\chi(v) h(v)\, dv = 0 \quad (96)$$

Since h depends linearly on $s_1 = v^{-1}s$, this is a linear constraint on s_1. When the constraint is satisfied, the argument proceeds exactly as before and we obtain the unique solution f.

If (96) is satisfied, then (1) has a solution (necessarily not unique). Let us integrate (1) over $dv\,dx$, taking into account (7) with $\alpha = 1$. We get

$$\int_0^\infty v g(v) e^{-v^2/2}\, dv + \int_{-\infty}^\infty dv \int_0^\infty dx\, \mathscr{S}(v,x) = 0$$

If we require physical (i.e., nonnegative) sources and incoming distributions, this relation implies $g = \mathscr{S} = 0$. We notice that the integrations above are allowed: the hypotheses in Theorem 1 that $g \in \mathscr{H}_T$ and $s_1 \in L^1(\mathbb{R}_+; \mathscr{H}_T)$ imply $v g(v) e^{-v^2/2} \in L^1(\mathbb{R}_+)$ and $\mathscr{S} \in L^1(\mathbb{R}_+ \times \mathbb{R})$.

7. THE TIME-DEPENDENT PROBLEM

The full time-dependent problem corresponding to the stationary problem above is

$$\frac{\partial f}{\partial v}(x,v,t) + v\frac{\partial f}{\partial x}(x,v,t) - \left[\left(\frac{\partial}{\partial v}\right)^2 - v\frac{\partial}{\partial v}\right] f(x,v,t) = 0, \qquad x,t > 0 \quad (97)$$

$$f(x,v,0) = f_0(x,v) \quad (98)$$

$$f(0,v,t) = \alpha f(0,-v,t), \qquad v,t > 0 \quad (99)$$

We write this problem as

$$\frac{\partial f}{\partial t} + Lf = 0, \qquad t > 0 \quad (100)$$

$$f|_{t=0} = f_0 \quad (101)$$

where L is the operator $T\partial/\partial x + A$ whose domain consists of those functions f from $[0, \infty]$ to $\mathscr{H}$ such that

$$f \in L^2(\mathbb{R}_+; \mathscr{H}) \cap C_0(\mathbb{R}_+; \mathscr{H}_T)$$
$$\frac{\partial f}{\partial v} \in L^2(\mathbb{R}_+; \mathscr{H}), \qquad Q_+(f(0, \cdot) - \alpha Rf(0, \cdot)) = 0 \tag{102}$$

and such that Lf, taken in the sense of distributions, belongs to $L^2(\mathbb{R}_+; \mathscr{H})$.

Existence and uniqueness of solutions of (100), (101) in a suitable sense will follow if we show that the operator $-L$ generates a contraction semigroup in the Hilbert space $L^2(\mathbb{R}_+; \mathscr{H}) = L^2(\mathbb{R}_+ \times \mathbb{R}, dx\, d\sigma(v))$, and this in turn will follow if we show that for $\lambda > 0$, $L + \lambda$ maps its domain onto $L^2(\mathbb{R}_+; \mathscr{H})$ and has an inverse with norm $\leqslant \lambda^{-1}$; see Ref. 17. To prove this amounts to considering the stationary problem with A replaced by $A + \lambda$, $\lambda > 0$. This operator is invertible, so we may repeat the previous arguments with $A + \lambda$ in place of A_1 and $S_\lambda = (A + \lambda)^{-1}T$ in place of S to conclude that for every suitable source function s there is a unique f in the domain of L such that $(L + \lambda)f = s$. In this case, however, the series of identities in (93) leads to the inequality

$$\int_0^\infty (\lambda f + Af, f)\, dx \leqslant \int_0^\infty (s, f)\, dx \tag{103}$$

Therefore

$$\lambda \int_0^\infty |f|^2\, dx \leqslant \left(\int_0^\infty |s|^2\, dx \int_0^\infty |f|^2\, dx \right)^{1/2} \tag{104}$$

and $(L + \lambda)^{-1}$ has norm $\leqslant \lambda^{-1}$ as operator in $L^2(\mathbb{R}_+; \mathscr{H})$.

Note that this argument remains valid in the purely reflecting case $\alpha = 1$. Roughly speaking, this means that when the initial distribution f_0 vanishes at $x = \infty$, then there is for all time a unique solution f which also vanishes at $x = \infty$ and satisfies the BC at $x = 0$, but when $\alpha = 1$ any constant can be added to the solution.

8. THE SLAB GEOMETRY

We conclude with a condensed discussion of the one-dimensional slab geometry. With a slab of width 1 the problem is

$$T\frac{\partial f}{\partial x} + Af = s, \qquad 0 < x < 1 \tag{105}$$

$$Q_+(f - \alpha Rf)|_{x=0} = Q_+ g$$
$$Q_-(f - \alpha Rf)|_{x=0} = Q_- g \tag{106}$$

Subtracting from f any solution of (105) alone, we reduce to

$$T\frac{\partial f}{\partial x}+Af=0,\qquad 0<x<1 \tag{107}$$

and the BC (106) with different g. It is convenient to begin with the same problem for $A_1 = A + P$ as above:

$$T\frac{\partial f}{\partial x}+A_1 f=0,\qquad 0<x<1 \tag{108}$$

Given $h \in \mathscr{H}_S$, define $f:[0,1]\to\mathscr{H}_T$ by

$$f(x) = U_1(x)h = e^{-xB}P_+ h + e^{(1-x)B}P_- h \tag{109}$$

where $B = S^{-1}$. Then f given by (109) solves (108), and conversely every solution of (108) has the form (109). To satisfy (106) we need

$$\begin{aligned} g &= Q_+\left[f(0) - \alpha Rf(0)\right] + Q_-\left[f(1) - \alpha Rf(1)\right] \\ &= V\left[(I - \alpha Re^{-|B|}) + V^{-1}W(e^{-|B|} - \alpha R)\right]h \\ &= V\Gamma_{1,\alpha}h \end{aligned} \tag{110}$$

Thus the question of existence and uniqueness of solutions to (107), (106) is the question where $\Gamma_{1,\alpha}$ is an isomorphism of $\mathscr{H}_S$ onto itself.

Note that $\|e^{-|B|}\| < 1$, where $\|\ \|$ denotes the operator norm in $\mathscr{H}_S$. Also, R is unitary in $\mathscr{H}_S$, so $I - \alpha Re^{-|B|}$ is invertible for $|\alpha| \leqslant 1$. Thus

$$\Gamma_{1,\alpha} = (I + V^{-1}WO_\alpha)(I - \alpha Re^{-|B|}) \tag{111}$$

$$O_\alpha = (e^{-|B|} - \alpha R)(I - \alpha Re^{-|B|})^{-1} \tag{112}$$

The adjoint of $V^{-1}W$ is $W'(V')^{-1}$ and the analog of (68) shows that this operator has norm < 1, so we have $\|V^{-1}W\| < 1$. Therefore we will have shown invertibility of $\Gamma_{1,\alpha}$ if we prove $\|O_\alpha\| \leqslant 1$. Recall that $Ru_n = u_{-n}$ and $R_-\varphi_+ = \varphi_-$. It follows that O_α is self-adjoint in $\mathscr{H}_S$ with two-dimensional invariant subspaces spanned by $\{u_n, u_{-n}\}$ and $\{\varphi_+, \varphi_-\}$. A direct calculation in each such subspace shows that the eigenvalues have absolute value $\leqslant 1$, so $\|O_\alpha\| \leqslant 1$ and $\Gamma_{1,\alpha}$ is invertible if $|\alpha| \leqslant 1$.

Returning to (107), we obtain a solution

$$f(x) = U(x)h = \alpha_0 + \alpha_1(v - x) + \sum_{n>0} a_n e^{-\lambda_n x}u_n + \sum_{n<0} a_n e^{\lambda_n(1-x)}u_n \tag{113}$$

where $h \in \mathscr{H}_S$ has the expansion

$$h = \alpha_0 + \alpha_1 v + \sum a_n u_n \tag{114}$$

Again the BC (106) may be expressed by means of an operator $\Gamma_\alpha : \mathscr{H}_S \to \mathscr{H}_T$. Now $U_1(x)h \equiv U(x)h$ if h lies in the closed linear span of $\{u_n\}$.

Therefore Γ_α differs from the invertible operator $\Gamma_{1,\alpha}$ by an operator of rank = 2, the dimension of the complement of this span. Then Γ_α is invertible if and only if it is one-to-one. But for a solution f of (107), (106) we have

$$\begin{aligned} 0 &\leqslant 2\int_0^1 (Af, f)\,dx = -2\int_0^1 \left(T\frac{\partial f}{\partial x}, f\right) dx = -\int_0^1 \frac{d}{dx}(Tf, f)\,dx \\ &= (1-\alpha^2)\int_{-\infty}^0 vf(0,v)^2\,dv - (1-\alpha^2)\int_0^\infty vf(1,v)^2 fv + |g|_T^2 \\ &\leqslant |g|_T^2, \qquad 0 \leqslant \alpha \leqslant 1 \end{aligned} \tag{115}$$

Thus $g = 0$ implies f constant, but for $\alpha \neq 1$ the only constant satisfying the BC is then 0. For $\alpha = 1$, any constant is allowed. Thus for $0 \leqslant \alpha < 1$, Γ_α is one-to-one and (107)–(106) has a unique solution. For $\alpha = 1$, Γ_α has range of codimension 1, so (107), (106) has a solution if and only if the data satisfy a single linear constraint; if so, then any constant may be added to the solution.

On a slab of length l the operator $e^{-|B|}$ in (110) is replaced by $e^{-l|B|}$. This operator tends exponentially to 0 as $l \to \infty$, so the solution of the slab problem tends rapidly to the solution of the problem for the semi-infinite medium.

Finally, we note that the time dependent problem for the slab may now be handled easily by the method of Section 7.

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APPENDIX: EQUIVALENCE OF THE NORMS $|\ |_S$ AND $|\ |_T$

It is sufficient to prove

$$|h|_T \leqslant m|h|_S, \qquad h \in \mathscr{H}_0 \tag{A.1}$$

In fact we may then deduce for $h \in \mathscr{H}_0$ that

$$\begin{aligned} |h|_S^2 &= (Sh, (P_+ - P_-)h)_A = (A_1 Sh, (P_+ - P_-)h) \\ &= (Th(P_+ - P_-)h) = ((Q_+ - Q_-)h, (P_+ - P_-)h)_T \\ &= |h|_T |(P_+ - P_-)h|_T \leqslant m|h|_T |(P_+ - P_-)h|_S = m|h|_T |h|_S \end{aligned}$$

and so $|h|_S \leqslant m|h|_T$. To prove (A.1) we use the following adaptation of a lemma of Baouendi and Grisvard.[18]

Lemma. There are linear operators X and Y, continuous as operators in $\mathscr{H}$ and in $\mathscr{H}_A$, such that

$$Xh(v) = h(v), \qquad v > 0, \quad h \in \mathscr{H} \tag{A.2}$$

$$|T|X = Y^*T \qquad \text{on } \mathscr{H}_A \tag{A.3}$$

where Y^* is the adjoint of Y as operator in $\mathscr{H}$.

There is of course a similar result with $v > 0$ in (A.2) replaced by $v < 0$. To prove the lemma we choose a cutoff function $\varphi \in C^1(\mathbb{R})$ with $\varphi(0) = 1$ and $\varphi(v) = 0$ for $|v| \geqslant 1$. Let X be defined by

$$\begin{aligned} Xh(v) &= h(v), && v \geqslant 0 \\ &= \alpha h(-v)\varphi(-v) + 4\beta h(-2v)\varphi(-2v), && v < 0 \end{aligned} \tag{A.4}$$

Here the constants α and β are to be determined. The requirement that $X : \mathscr{H}_A \to \mathscr{H}_A$ will be fulfilled if $\alpha + 4\beta = 1$. With this condition, X will be continuous. To satisfy (A.4) we need

$$\begin{aligned} Y^*h(v) &= h(v), && v > 0 \\ Y^*h(v) &= \alpha h(-v)\varphi(-v) + 2\beta h(-2v)\varphi(-2v), && v < 0 \end{aligned} \tag{A.5}$$

An easy computation shows that the adjoint Y is given by

$$\begin{aligned} Yh(v) &= 0, && v < 0 \\ Yh(v) &= h(v) + \alpha\varphi(v)h(-v) + \beta\varphi_1(v)h(-\tfrac{1}{2}v), && v > 0 \end{aligned} \tag{A.6}$$

where $\varphi_1(v) = \varphi(v)\exp[\frac{1}{2}v^2 - \frac{1}{8}v^2]$. For Y to map $\mathscr{H}_A$ to $\mathscr{H}_A$ we need $1 + \alpha + \beta = 0$. Thus we prove the lemma by taking $\alpha = -5/3$, $\beta = 2/3$.

To prove (A.1) it is enough to consider the two cases: $h \in \mathscr{H}_A$ and $P_+ h = h$ or $P_- h = h$. We assume $P_+ h = h$. The range of S is dense, so we may assume $h \in S(\mathscr{H}_A)$. Define the $\mathscr{H}_A$-valued function u by

$$u(t) = e^{-tS^{-1}}h, \qquad t \geqslant 0 \tag{A.7}$$

This makes sense since $P_+ h = h$ (for $P_- h = h$ we take $t \leqslant 0$); moreover $u(t)$ is in the range of S for all $t \geqslant 0$, the function u is strongly differentiable in $\mathscr{H}_A$, and

$$T\frac{du}{dt} = -TS^{-1}u = -A_1 u \tag{A.8}$$

Also, $|u(t)|_A \to 0$ as $t \to \infty$.

We want to estimate

$$|h|_T^2 = \int |v| h(v)^2 \, dv$$

We estimate the integral over $v > 0$; the other estimation uses the other form of the lemma. Using (A.2) we have

$$\int_0^\infty vh(v)^2\,d\sigma(v) \leqslant |Xh|_T^2$$

Now as $t \to \infty$,

$$|Xu(t)|_T^2 \leqslant |TXu(t)|\,|Xu(t)| \leqslant C|u(t)|_A^2 \to 0$$

Therefore

$$\begin{aligned}
\frac{1}{2}|Xh|_T^2 &= -\frac{1}{2}\int_0^\infty \frac{d}{dt}|Xu(t)|_T^2\,dt \\
&= -\int_0^\infty (|T|Xu'(t), Xu(t))\,dt = -\int_0^\infty (Y^*Tu'(t), Xu(t))\,dt \\
&= \int_0^\infty (A_1u(t), YXu(t))\,dt = \int_0^\infty (u, YXu)_A\,dt \\
&\leqslant C\int_0^\infty |u|_A^2\,dt = C\int_0^\infty (e^{-2tS^{-1}}h, h)_A\,dt = \frac{1}{2}C(Sh, h)_A = \frac{1}{2}C|h|_S^2
\end{aligned}$$

we have used the identity

$$\int_0^\infty e^{-tB}\,dt = B^{-1}$$

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